INFO 295, 19 Oct 04

To prove Stirling's formula, $N! \sim \sqrt{2\pi N} N^N \mathrm{e}^{-N}$ for large N, we first recall the method of steepest descent. An integral of the form,

$$I = \int_a^b \mathrm{e}^{N f(x)} \mathrm{d}x \ ,$$

when N is large, is dominated by the "critical points" x_0 of f, at which $f'(x_0) = 0$. At any such point, we can approximate in the near vicinity of x_0,

$$f(x) \approx f(x_0) - \frac{1}{2} |f''(x_0)| (x - x_0)^2 \ . \tag{1}$$

Assuming only a single critical point, we can then approximate

$$I \approx \mathrm{e}^{N f(x_0)} \int_{-\infty}^{\infty} \mathrm{e}^{-\frac{1}{2} N |f''(x_0)| x^2} \, \mathrm{d}x = \mathrm{e}^{N f(x_0)} \sqrt{\frac{2\pi}{N |f''(x_0)|}}$$

(where we have also extended the limits of integration from $-\infty$ to ∞, assuming that only the region near x_0 matters anyway, and then used $\int_{-\infty}^{\infty} \mathrm{d}x \, \mathrm{e}^{-ax^2} = \sqrt{\pi/a}$ [†]).

To apply to the factorial function, we first need an integral representation. The function

$$\Gamma(N+1) = \int_0^\infty \mathrm{e}^{-x} \, x^N \mathrm{d}x$$

satisfies the recursion relation

$$\Gamma(N+1) = -\mathrm{e}^{-x} \, x^N \Big|_0^\infty + N \int_0^\infty \mathrm{e}^{-x} \, x^{N-1} \, \mathrm{d}x = N \, \Gamma(N)$$

(using integration by parts). Together with the boundary condition $\Gamma(1) = \int_0^\infty \mathrm{e}^{-x} \mathrm{d}x = 1$, we find that $\Gamma(N+1) = N!$ for integer N.

So we write

$$N! = \Gamma(N+1) = N^{N+1} \int_0^\infty \mathrm{e}^{N(\ln z - z)} \mathrm{d}z$$

where $z = x/N$. Hence $f = \ln z - z$, $f' = 1/z - 1$, $f'' = -1/z^2$, and $z_0 = 1$. Eqn. (1) gives

$$N! \approx N^{N+1} \, \mathrm{e}^{-N} \sqrt{\frac{2\pi}{N}} = \sqrt{2\pi N} \, N^N \mathrm{e}^{-N}$$

[†] If we write $J = \int_{-\infty}^{\infty} \mathrm{d}x \, \mathrm{e}^{-x^2}$, then using r, θ cylindrical coordinates its square can be written $J^2 = \int_{-\infty}^{\infty} \mathrm{d}x \int_{-\infty}^{\infty} \mathrm{d}y \, \mathrm{e}^{-x^2 - y^2} = \int_0^{2\pi} \mathrm{d}\theta \int_0^\infty \mathrm{d}r \, r \, \mathrm{e}^{-r^2} = 2\pi \Big(-\frac{1}{2} \mathrm{e}^{-r^2} \Big|_0^\infty \Big) = \pi$. Thus $J = \sqrt{\pi}$ and $\int_{-\infty}^{\infty} \mathrm{d}x \, \mathrm{e}^{-ax^2} = \sqrt{\pi/a}$.